MATH 312

Lecture 27.

THE FOURIER TRANSFORM

What we will need Today

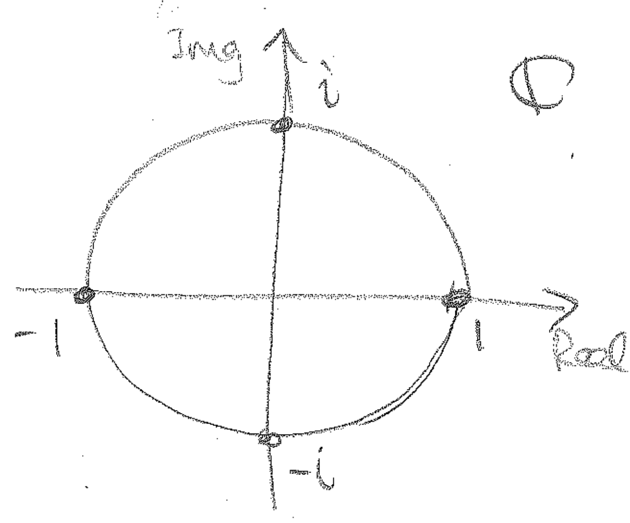
ORTHOGONALITY: $u \perp v \Leftrightarrow u^T v = 0$.

CALCULUS: $\int \cos$, $\int \sin$, $\int e^{\cdot}$, Power series for each...

only a little...

And, a crash course on
THE COMPLEX ROOTS OF UNITY.

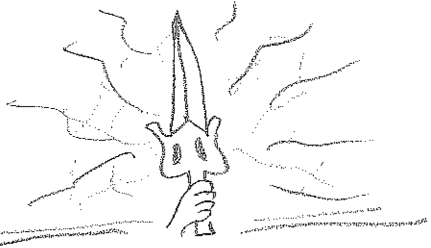
- When working with Real numbers x, the equation
$$\boxed{x^n = 1}$$
has only ONE solution for <u>odd</u> n, and two for <u>even</u> n.
- But over the Complex numbers, it <u>ALWAYS</u> has n distinct solutions! Start with
$$\boxed{z^4 = 1}$$
and note that ± 1 and $\pm i$ solve this!
- Pictorially, they also happen to quadrisect the unit circle sitting in the complex plane!



The 4-th roots of 1 in $\mathbb{C}$.

- <u>QUESTION</u>: How to solve $z^n = 1$ for arbitrary n's?

For an answer, we turn to Euler!



"i" has the power!

$i^0 = 1$
$i^1 = i$
$i^2 = -1$
$i^3 = -i$
$i^4 = 1$
(repeat)

Prop $$e^{i\theta} = \cos\theta + i\sin\theta.$$

Pf Power series!

$$e^x = \sum_{k=0}^{\infty} x^k/k!$$

So, $$e^{i\theta} = \sum_{k=0}^{\infty} (i\theta)^k/k!$$

Separate "real" and "complex" stuff

All even powers of i are real and odd ones imaginary

So,

$$e^{i\theta} = \underbrace{\left(\sum_{k \text{ even}} (-1)^k \theta^k/k!\right)}_{\text{Power series for } \cos\theta} + i \underbrace{\left(\sum_{k \text{ odd}} (-1)^k \theta^k/k!\right)}_{\text{Power series for } \sin\theta}$$

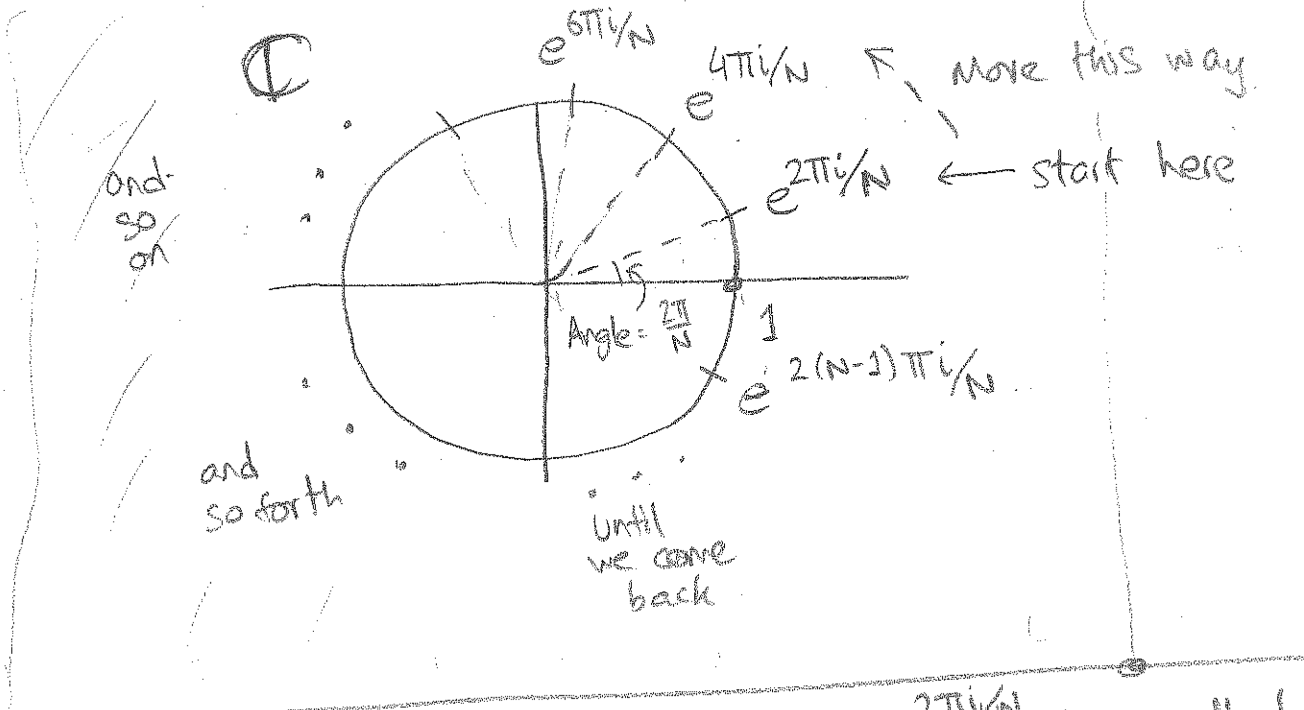
$$= \cos\theta + i\sin\theta$$

Now, If $z = e^{2\pi i/N}$, then $z^N = e^{2\pi i} = 1$

Here, we are writing complex numbers in POLAR FORM:

$x + iy$ =	$re^{i\theta}$, where
$x = r\cos\theta$	$r^2 = x^2 + y^2$
$y = r\sin\theta$	$\theta = \arctan(y/x)$

So, the N-th roots of UNITY look like this:



The "first" of these roots $\omega_N = e^{2\pi i/N}$ is called the PRIMITIVE root of 1, all others are its "POWERS":

$$\left\{ \underset{\substack{\| \\ 1}}{\omega_N^0}\ \ \underset{\substack{\| \\ e^{2\pi i/N}}}{\omega_N^1},\ \underset{\substack{\| \\ e^{4\pi i/N}}}{\omega_N^2},\ \underset{\substack{\| \\ e^{6\pi i/N}}}{\omega_N^3},\ \cdots,\ \underset{\substack{\| \\ e^{2(N-2)\pi i/N}}}{\omega_N^{N-2}},\ \underset{\substack{\| \\ e^{2(N-1)\pi i/N}}}{\omega_N^{N-1}} \right\}$$

Prop $(N>1)$ Let ω_N be the primite N-th root of unity. Then,

$$\boxed{\omega_N^0 + \omega_N^1 + \cdots + \omega_N^{N-1} = 0}$$

Pf Multiply left side by $(1-\omega_N)$!

Okay, that's all the BACKGROUND we will need.

Today's mission is to produce the following:

LINEAR ALGEBRA IN INFINITE DIMENSIONS !!!

CASE I | ℓ_2 - space

Since $\mathbb{C}$ is just a collection of n-tuples of complex numbers, one can define "sequences":

- $\mathbb{C}^\infty = \{(x_1, x_2, \ldots)$ with each x_i in $\mathbb{C}\}$

never-ending.

This is a vector space (addition, scaling are easy to define!) BUT we have no "lengths" for general vectors $\vec{x} = (x_1, x_2, \ldots)$ because

$\boxed{\bar{x} \text{ denotes complex conjugate!}}$

$$\|X\| = (x_1 \cdot \bar{x}_1 + x_2 \cdot \bar{x}_2 + \cdots).$$

Why should this even be finite?

- Eg: the vector with all 1's has infinite length...

There's no hope in general, so let's look at a small piece of $\mathbb{C}$: "vectors with finite length".
This is the space $\ell_2(\mathbb{R})$: or, $(x_1^2 + x_2^2 + \cdots)$ converges.

Def $\ell_2(\mathbb{C}) = \{\vec{x}$ in $\mathbb{C}^\infty$ so that $\|\vec{X}\| < \infty\}$

- Note: $\ell_2(\mathbb{C})$ contains a copy of $\mathbb{C}$ for every n!!

- A "basis" for $\ell_2(\mathbb{C})$ is $\underbrace{(1, 0, \ldots)}_{e_1}, \underbrace{(0, 1, 0, \ldots)}_{e_2}$ etc.

- Angles are defined: $x^T \bar{y} = x_1\bar{y}_1 + \cdots + x_n\bar{y}_n + \cdots$ is ALWAYS smaller than $\|X\| \cdot \|y\|$.

CASE II | L_2 - space

- Think of vectors in $\mathbb{C}^n$ as functions on $\{1, \dots, n\}$

$$\vec{v} : \{1, 2, \dots, n\} \to \mathbb{C}, \quad \vec{v}(1) = v_1, \text{ etc.}$$

first coordinate

- And in $\mathbb{C}^\infty$ as functions on $\mathbb{N}$:

$$\vec{v} : \mathbb{N} \to \mathbb{C}, \quad \text{coordinates go forever.}$$

- Why stop there?

$$\mathbb{C}^{\mathbb{C}} = \{\text{all functions } f : \mathbb{C} \to \mathbb{C}\}$$

Now, "lengths" are given by integrals, rather than sums:

$$\|f\| = \left[\int_{-\infty}^{\infty} f(t) \cdot \bar{f}(t)\, dt \right]^{1/2}$$

But again, this may be ∞. So, we restrict to $[0, 2\pi]$:

Def $L_2[0, 2\pi]$ is the space of all $f : [0, 2\pi] \to \mathbb{C}$ for which the integral $\int_0^{2\pi} f(t) \cdot \bar{f}(t)\, dt < \infty$.

Again, this is a vector space: we can add and scale functions! But the SURPRISING fact about L_2 is:

Prop The functions $\{ e^{int} : n = 0, 1, 2, \dots \}$ form an orthogonal basis for $L^2[0, 2\pi]$.

Pf There are two things to check:

1. $\{e^{int}\}$ span L_2
2. $\{e^{int}\}$ are orthogonal (hence independent)

"1" is difficult: it involves a result called the STONE-WEIERSTRASS Theorem. But 2. is easy! What passes for "dot product" $u^T\bar{v}$ in L_2 is an integral:

$$(e^{int})^{\overset{\text{transpose}}{T}} \cdot \overline{e^{imt}}^{\text{conjugate}} = \int_0^{2\pi} e^{int} \cdot e^{-imt}\, dt = \int_0^{2\pi} e^{i(n-m)t}\, dt$$

If $m=n$, then we get $\int_0^{2\pi} e^0 dt = \underline{2\pi}$.

If $m \neq n$, then the integral evaluates to

$$\int_0^{2\pi} e^{i(n-m)t}\, dt = \frac{1}{i(n-m)} \left[e^{i(n-m)t} \right]_{t=0}^{t=2\pi}$$

But $e^0 = 1 = e^{2\pi i(n-m)}$, so this part equals zero!!

So: $$(e^{int})^T \cdot \overline{(e^{imt})} = \begin{cases} 2\pi, & m=n \\ 0, & \text{else.} \end{cases}$$

Thus $\{e^{int}\}$ is orthogonal.

NOTE | There are two EXTREMELY IMPORTANT consequences

a) EVERY f in $L_2[0,2\pi]$ can be written as

$$f(t) = \sum_{n=0}^{\infty} a_n \cdot e^{int}$$ for some coefficients a_n in $\mathbb{C}$.

i.e., $f(t) = a_0 e^{i\cdot 0\cdot t} + a_1 e^{i\cdot 1\cdot t} + a_2 e^{i\cdot 2\cdot t} + \ldots$ going forever.

b) to determine the j-th coefficient a_j for your f, just use orthogonality of the basis:

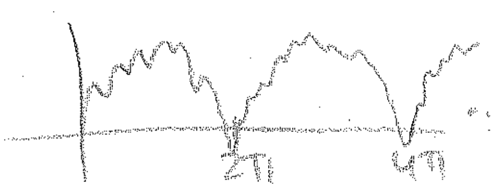
$$\boxed{a_j = \frac{1}{2\pi}\int_0^{2\pi} f(t)\, e^{-ijt}\, dt}$$

And now we are <u>Ready</u>:

<u>Def</u> The $\boxed{\text{FOURIER TRANSFORM}}$ is a correspondence between $L^2[0,2\pi]$ and $\ell_2(\mathbb{C})$! Given any f in $L^2[0,1]$, we produce a sequence $(a_0, a_1, \dots)$ in ℓ_2 by

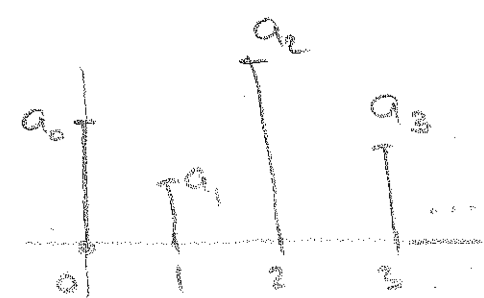
$$\boxed{a_j = \frac{1}{2\pi}\int_0^{2\pi} f(t)\, e^{-ijt}\, dt}$$

$\mathcal{F}$



A 2π-periodic function f(t)

fourier transform



a sequence $(a_0, \dots)$

$L_2[0,2\pi]$

$\mathcal{F}^{-1}$

$\ell_2(\mathbb{C})$

"Time domain"

inverse Fourier: we will see this later

"Frequency domain"